# Span and Chainability in Non-metric Continua

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DEFINITION A continuum $X$ is chainable if every open cover has an open cover refinement which is a chain.

# Span

DEFINITION A continuum $X$ has span zero if every subcontinuum $Z$ of $X \times X$, which projects onto the same set on both coordinates, has a nonempty intersection with the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$ of $X$. Otherwise we say that $X$ has span non-zero.

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OUR RESULT If there is a non-metric counterexample, there is also a metric counterexample.

# Wallman's representation theorem

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$$\forall ab\ \exists c\ (a \not\leq b \rightarrow c \neq 0 \text{ and } c \leq a \text{ and } b \wedge c = 0).$$

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DEFINITION A lattice is called normal if it models the following sentence

$$\forall ab\ \exists cd\ (a \wedge b = 0 \to a \wedge d = 0 \text{ and } b \wedge c = \mathbf{0} \text{ and } c \vee d = 1).$$

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Wallman's representation extends to lattice homomorphisms and provides a functor $w$.

# Ultracopower

DEFINITION $q : X \times I \to I$ projection, $I$ discrete
$\beta(q) : \beta(X \times I) \to \beta(I)$ - Čech-Stone lifting of $q$
Ultracopower $\sum_{\mathcal{U}} X$ of $X$ with respect to an ultrafilter $\mathcal{U}$ on $I$ is $(\beta(q))^{-1}[\mathcal{U}]$.

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LEMMA Let $\Delta : B \to \prod_{\mathcal{U}} B$ be the diagonal embedding of a distributive disjunctive normal lattice $B$ to its ultrapower. Then $w(\Delta) = \nabla$.

# Elementarity

Fix a first-order language $\mathcal{L}$.

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DEFINITION $A$ and $B$ - $\mathcal{L}$-structures. $B$ is an elementary substructure of $A$ if $B$ is a substructure of $A$ and for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots a_n \in B$

$$B \models \phi[a_1, \ldots, a_n] \text{ if and only if } A \models \phi[a_1, \ldots, a_n].$$

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LÖWENHEIM-SKOLEM THEOREM Let $A$ be an infinite $\mathcal{L}$-structure and let $X \subset A$. Denote $\kappa = \max(|\mathcal{L}|, |X|)$. Then for every cardinal $\lambda$ such that $\kappa \leq \lambda \leq |A|$, there exists an elementary substructure $B$ of $A$ such that $X \subset B$ and $|B| = \lambda$.

# Elementarity in set theory

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These sets are very important and useful because if $\theta$ is uncountable regular then

$$H(\theta) \models \text{ ZFC - P }.$$

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These sets are very important and useful because if $\theta$ is uncountable regular then

$$H(\theta) \models \text{ ZFC - P } .$$

If $\mathcal{M}$ is an elementary submodel of $H(\theta)$ such that $2^X \in \mathcal{M}$ then $L = \mathcal{M} \cap 2^X$ is an elementary sublattice of $2^X$. Similarly $K = \mathcal{M} \cap 2^{X \times X}$ is an elementary sublattice of $2^{X \times X}$.

# Applying elementarity

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THEOREM (van der Steeg 2003) $X$ is chainable if and only if $wL$ is chainable.

# Keisler-Shelah theorem

KEISLER-SHELAH THEOREM Let $\kappa$ be a cardinal, $\lambda = \min\{\mu \mid \kappa^\mu > \kappa\}$ and let $A$ and $B$ be two elementarily equivalent $\mathcal{L}$-structures with $\operatorname{card}(A), \operatorname{card}(B) < \lambda$. Then there exists an ultrafilter $\mathcal{U}$ over $\kappa$ such that $\prod_{\mathcal{U}} A$ and $\prod_{\mathcal{U}} B$ are isomorphic.

# Reflecting span zero

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Proof

$$\begin{array}{ccc} K & \xrightarrow{\ e\ } & 2^{X \times X} \\ \Big\downarrow \Delta & & \Big\downarrow \Delta \\ \prod_{\mathcal{U}} K & \xrightarrow{h} & \prod_{\mathcal{U}} 2^{X \times X} \end{array} \tag{1}$$

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$$
\begin{array}{ccc}
wL \times wL \cong wK & \xleftarrow{\;w(e)\;} & X \times X \\
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$$Z' = \nabla \circ w(h)^{-1}[\textstyle\sum_{\mathcal{U}} Z].$$

# Questions

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Question 2 If $L$ is an elementary sublattice of $2^X$, is the Wallman representation of the elementary embedding of $L$ into $2^X$ confluent.

Question 3 Is there a (non)-metric continuum that has span zero and is not chainable?

THANK YOU!!!