Span and Chainability in Non-metric Continua

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Chainability

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DEFINITION Let X be a continuum. A chain is a nonempty, finite collection $\mathcal{C} = \{C_1, \ldots, C_n\}$ of open subsets C_i of X such that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The elements C_i of \mathcal{C} are called links of the chain \mathcal{C} .

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DEFINITION A continuum X is chainable if every open cover has an open cover refinement which is a chain.

Span

DEFINITION A continuum X has span zero if every subcontinuum Z of $X \times X$, which projects onto the same set on both coordinates, has a nonempty intersection with the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$ of X. Otherwise we say that X has span non-zero.

Lelek's conjecture

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OUR RESULT If there is a non-metric counterexample, there is also a metric counterexample.

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$$\forall ab \; \exists c \; (a \nleq b \rightarrow c \neq 0 \; \text{and} \; c \leq a \; \text{and} \; b \land c = 0).$$

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DEFINITION A lattice is called normal if it models the following sentence

$$\forall ab \; \exists cd \; (a \land b = 0 \rightarrow a \land d = 0 \; \text{and} \; b \land c = \mathbf{0} \; \text{and} \; c \lor d = 1).$$

THEOREM (Wallman 1938) Let L be a distributive disjunctive normal lattice. Then there is a compact Hausdorff space wL with a base for closed sets being isomorphic to L.

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$$X \rightarrow \mathcal{B} \rightarrow w\mathcal{B} = X$$
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Wallman's representation extends to lattice homomorphisms and provides a functor w.

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DEFINITION q: X \times I \to I projection, I discrete \beta(q): \beta(X \times I) \to \beta(I) - Čech-Stone lifting of q Ultracopower \sum_{\mathcal{U}} X of X with respect to an ultrafilter \mathcal{U} on I is (\beta(q))^{-1}[\mathcal{U}].
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LEMMA Let $\Delta: B \to \prod_{\mathcal{U}} B$ be the diagonal embedding of a distributive disjunctive normal lattice B to its ultrapower. Then $w(\Delta) = \nabla$.

Elementarity

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DEFINITION A and B - \mathcal{L} -structures. B is an elementary substructure of A if B is a substructure of A and for every formula $\phi(x_1,\ldots,x_n)$ and $a_1,\ldots a_n\in B$

$$B \models \phi[a_1, \ldots, a_n]$$
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LÖWENHEIM-SKOLEM THEOREM Let A be an infinite \mathcal{L} -structure and let $X\subset A$. Denote $\kappa=\max(|\mathcal{L}|\,,|X|)$. Then for every cardinal λ such that $\kappa\leq\lambda\leq|A|\,,$ there exists an elementary substructure B of A such that $X\subset B$ and $|B|=\lambda$.



Elementarity in set theory

For a cardinal θ , $H(\theta)$ denotes the set of all sets whose transitive closure has cardinality less then θ .

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If \mathcal{M} is an elementary submodel of $H(\theta)$ such that $2^X \in \mathcal{M}$ then $L = \mathcal{M} \cap 2^X$ is an elementary sublattice of 2^X . Similarly $K = \mathcal{M} \cap 2^{X \times X}$ is an elementary sublattice of $2^{X \times X}$.

Applying elementarity

THEOREM (van der Steeg 2003) $wK \cong wL \times wL$

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THEOREM (van der Steeg 2003) X is chainable if and only if wL is chainable.

Keisler-Shelah theorem

KEISLER-SHELAH THEOREM Let κ be a cardinal, $\lambda = \min\{\mu \mid \kappa^{\mu} > \kappa\}$ and let A and B be two elementarily equivalent \mathcal{L} -structures with $\operatorname{card}(A), \operatorname{card}(B) < \lambda$. Then there exists an ultrafilter \mathcal{U} over κ such that $\prod_{\mathcal{U}} A$ and $\prod_{\mathcal{U}} B$ are isomorphic.

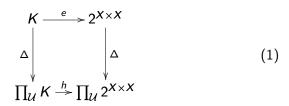
Reflecting span zero

THEOREM (DB+KPH 2008) If X is a continuum having span zero, then wL has span zero as well.

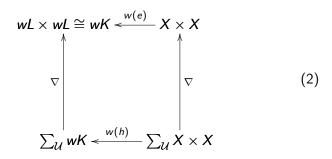
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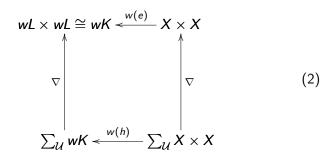
Proof



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$$Z' = \nabla \circ w(h)^{-1}[\sum_{\mathcal{U}} Z].$$



Questions

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Question 2 If L is an elementary sublattice of 2^X , is the Wallman representation of the elementary embedding of L into 2^X confluent.

Question 3 Is there a (non)-metric continuum that has span zero and is not chainable?

THANK YOU!!!